16.8 Stokes Theorem

Relationship between Surface integral, over a Surface S to a line integral around boundary Curve of S (Simply Closed) (Space Curve).

Define Positive Orientation of C.

Depends on the orientation of S (where normal vectors of S are directed)

Thm

- S oriented piecewise smooth surface
- S bounded by a simple, closed, picewise smooth boundary curve C with positive orientation.
- $\vec{F}(x,y,z)$ vector field whose components have conts partial derivatives in an open region $R \subset \mathbb{R}^3$ s.t. $S \subset R$,

Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}.$$

Notice that

$$\int_{\mathcal{C}} F.\, d\vec{r} = \iint_S \nabla \times \vec{F}.\, d\vec{S}$$

is equivalent to

$$\int_{\mathcal{C}} \vec{F}.\hat{T}\, ds = \iint_S \nabla \times \vec{F}.\, \hat{n}\, dS,$$

$\hat{T}$: unit tangent vector to $\mathcal{C}$

$\hat{n}$: unit normal vector to S.

which is also equivalent to

$$\int_a^b \vec{F}(\vec{r}(t)).\, \vec{r}\,'(t)\,dt = \iint_D \nabla \times \vec{F}(\vec{r}(u,v)).\,(\vec{r}_u \times \vec{r}_v)(u,v)\, du\,dv$$

where

$\mathcal{C}$: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $\quad a \le t \le b$

S: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $\quad (u,v) \in D$.

Section 16.8

Exercise 12 Find $\int_{C} \vec{F}\cdot d\vec{r}$, where for $\vec{F}(x,y,z)=\langle x^2y,\ \frac{1}{3}x^3,\ xy\rangle$

and C is the curve of intersection of

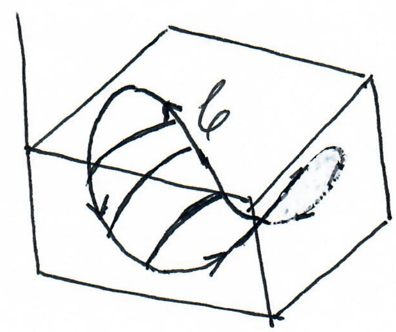
$$\begin{cases} S_1: & z = y^2 - x^2 \\ S_2: & x^2+y^2=1 \end{cases}$$

oriented counterclockwise as viewed from above.

Answ: A parametrization of C is

$$C: \quad \vec{r}(t)=\langle x(t), y(t), z(t)\rangle = \langle \cos t, \sin t, \sin^2 t - \cos^2 t\rangle$$
$$0 \le t \le 2\pi$$

Then,



$$\int_{C} \vec{F}\cdot d\vec{r} =$$

$$\int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\, dt$$

$$\int_0^{2\pi} \left\langle \cos^2 t \sin t, \frac{1}{3}\cos^3 t, \cos t \sin t\right\rangle \cdot \left\langle -\sin t, \cos t, \underbrace{2\sin t\cos t + 2\cos t \sin t}_{4\sin t \cos t}\right\rangle dt$$

$$= \int_0^{2\pi} \left(-\cos^2 t \sin^2 t + \frac{1}{3}\cos^4 t + 4\sin^2 t \cos^2 t\right) dt$$

$$= \int_0^{2\pi} \left(3\sin^2 t \cos^2 t + \frac{1}{3}\cos^4 t\right) dt = \text{Not trivial}$$

Easier evaluation of this integral can be performed using Stoke's Theorem.

$$\int_{C} \vec{F}\cdot d\vec{r} = \iint_{S} \nabla\times\vec{F}\, d\vec{S}.$$

First, we realize that for S oriented upward C is positively oriented if C is traversed counterclockwise.

In order to compute the Surface integral $\iint_S \nabla\times\vec{F}\,d\vec{S}$, we need to parametrize S. This can be done as follows:

$$S:\ \vec{r}(u,v) = \langle \overset{x}{u}, \overset{y}{v}, \overset{z}{v^2-u^2}\rangle, \quad (u,v)\in D = \{(u,v)\mid u^2+v^2\le 1\}$$

or $\quad -1\le u\le 1$

$\quad -\sqrt{1-u^2}\le v\le \sqrt{1-u^2}$

Then, $$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & 2v \end{vmatrix} = \langle 2u, -2v, 1\rangle$$

S is upward oriented.

We also need

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & x^3/3 & xy \end{vmatrix} = \langle x-0, -y+0, x^2-x^2\rangle = \langle x, -y, 0\rangle.$$

In the previous section 16.7, we already computed the integral and we arrived $\iint_S \nabla\times\vec{F}\,d\vec{S} = \pi$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & 2v \end{vmatrix} = \langle 2u, -2v, 1 \rangle$$

Upward normal

Then,

$$\iint_S \vec{G}(s)\,d\vec{s} = \iint_D \vec{G}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\,dA$$

$$= \iint_{D} \langle u, -v, 0 \rangle \cdot \langle 2u, -2v, 1 \rangle\,dA$$

D Disk of radius = 1

$$= \iint_D (2u^2 + 2v^2)\,dA = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (2u^2 + 2v^2)\,dv\,du$$

Cylind. coords $\xrightarrow{D}$ $u = r\cos\theta$, $v = r\sin\theta$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$$= \int_0^{2\pi} \int_0^1 (2r^2\cos^2\theta + 2r^2\sin^2\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 2r^2(\underbrace{\cos^2 + \sin^2}_{=1})\,r\,dr\,d\theta$$

$$= 2\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta = \frac{1}{2}\,2\pi = \pi$$

In the previous example,

S: Portion of $\underbrace{\text{Hyperbolic paraboloid}}$ bounded by Cylinder

$$\boxed{\begin{array}{l} z = y^2 - x^2 \\ x^2 + y^2 = 1 \end{array}}$$

Its parametrization in terms of x and y

$S: \vec{r}(x,y) = \langle x, y, y^2 - x^2 \rangle$, for x, y satisfying

$0 \leq x^2 + y^2 \leq 1$

Dissc of radius 1.

Then,

$$\vec{n} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & 2y \end{vmatrix} = \langle 2x, -2y, 1 \rangle$$

In general, if Surface S is defined as a function of x and y

$$z = f(x,y), \quad (x,y) \in D.$$

then a possible parametrization is: $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$(x,y) \in D$.

and

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \partial f/\partial x \\ 0 & 1 & \partial f/\partial y \end{vmatrix} = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

16.8 #3) $\vec{F}(x,y,z) = \langle z e^y, x\cos y, xz\sin y \rangle$

$S: \quad x^2+y^2+z^2=16, \quad y \geq 0$

oriented in the direction of positive y-axis.

Compute

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S}.$$

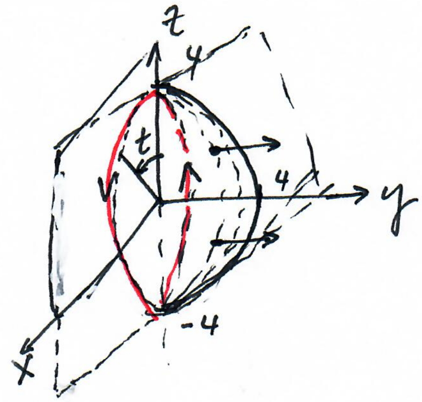
<u>Answ:</u>

Intersection

$\mathcal{C}: \; x^2+z^2=16, \quad y=0$

or $\hat{r}(t) = \langle 4\sin t, 0, 4\cos t \rangle$

$0 \le t \le 2\pi$



Then,

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \overset{ze^y}{4\cos t}, \overset{x\cos y}{4\sin t\,(1)}, \overset{xz\sin y}{0} \rangle \cdot \langle 4\cos t, 0, -4\sin t \rangle dt$$

$$= \int_0^{2\pi} 16\cos^2 t \, dt = \frac{16}{2}\int_0^{2\pi} 1\,dt + \frac{16}{2}\int_0^{2\pi} \cos 2t = \frac{16}{2}(2\pi) = 16\pi.$$

Since $\displaystyle\int_0^{2\pi} \cos 2t\,dt = \frac{\sin 2t}{2}\Big|_0^{2\pi} = \frac{1}{2}(\sin 4\pi - \sin 0) = 0.$

$\cos^2 t = 1 - \sin^2 t$

$\cos 2t = 1 - 2\sin^2 t \Rightarrow \sin^2 t = \dfrac{1-\cos 2t}{2}$

$\sin^2 t = 1 - \cos^2 t$

$\cos 2t = 2\cos^2 t - 1$

$\Rightarrow \cos^2 t = \dfrac{1+\cos 2t}{2}$

<u>Alternative:</u>

$S:$ xz-plane or $\underline{y=0}$

$\vec{r}(x,z) = \langle x, 0, z \rangle, \quad (x,z) \in D$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

$0 \le x^2 + z^2 \le 4$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ze^y & x\cos y & xz\sin y \end{vmatrix} =$$

$$= \langle xz\cos y,\ e^y - z\sin y,\ \cos y - ze^y \rangle$$

$$\hat{n} = \langle 0,1,0 \rangle = r_z \times r_x$$

Then,

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_{D\ \text{disk } r=4} \nabla \times \vec{F}(x,0,z) \cdot \langle 0,1,0 \rangle \, dx\,dy$$

$$= \iint_D \langle xz, 1, 1-z \rangle \cdot \langle 0,1,0 \rangle \, dx\,dy$$

$$= \iint_{\substack{D \\ 0 \le x^2+y^2 \le 4}} dx\,dy = \pi(4)^2 = \underline{\underline{16\pi}}$$

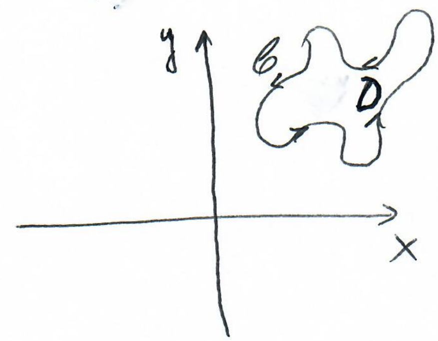
Green's theorem is a particular case of Stoke's theorem

Since for

$\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$

And $C: \vec{r}(t) = \langle x(t), y(t), 0 \rangle$ Simple closed
$a \le t \le b$

bounding a plane surface D, defined parametrically as



$D: \vec{r}(u,v) = \langle x(u,v), y(u,v), 0 \rangle$

$(u,v) \in R$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P(x(t),y(t)), Q(x(t),y(t)), 0 \rangle \cdot \langle x'(t), y'(t), 0 \rangle \, dt$$

$$= \int_a^b P(x(t),y(t))\, x'(t)\,dt + Q(x(t),y(t))\, y'(t)\,dt$$

$$= \int P\,dx + Q\,dy.$$

Also,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \langle 0, 0, x_u y_v - y_u x_v \rangle$$

$$= (x_u y_v - y_u x_v)\hat{k}$$

For C to be positively oriented

$\vec{n}$ should be positive. (out of the page)

or $\vec{n} = |\vec{r}_u \times \vec{r}_v|\hat{k} = |x_u y_v - y_u x_v|\hat{k}$.

Also,

$$\nabla \times \vec{F}(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \langle 0-0, 0-0, Q_x - P_y \rangle$$

Thus,

$$\int_C P\,dx + Q\,dy = \int_C \vec{F} \cdot d\vec{r} \overset{\text{Stoke's Thm.}}{=} \iint_{D} \nabla \times \vec{F} \cdot d\vec{S} =$$

upward = out of page

$$\iint_R \nabla \times \vec{F}(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v|\hat{k}\, du\, dv =$$

$$= \iint_R [Q_x(x(u,v), y(u,v)) - P_y(x(u,v), y(u,v))]\hat{k} \cdot |x_u y_v - y_u x_v|\hat{k}$$

$$= \iint_R [Q_x(x(u,v), y(u,v)) - P_y(x(u,v), y(u,v))] \Big(\underbrace{|x_u y_v - y_u x_v|}_{\text{Jacobian of transformation } (u,v) \to (x,y)} \Big) du\,dv$$

Changing Variables to x and y

$$\iint_D (Q_x(x,y) - P_y(x,y))\, dx\,dy = \iint_D (Q_x - P_y)\, dA.$$